Extremal eigenvalues of sparse random matrices

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Definition. *Wigner matrices* are $N \times N$ symmetric matrices $W = (W_{ij})_{ij}$ such that,

$$\mathbb{E}[W_{ij}] = 0, \qquad \mathbb{E}[(W_{ij})^2] = \frac{1}{N}$$

and $\{W_{ij}\}_{i \leqslant j}$ are iid.

An important example is when W_{ij} are all Gaussian. This is the Gaussian Orthogonal Ensemble (GOE).

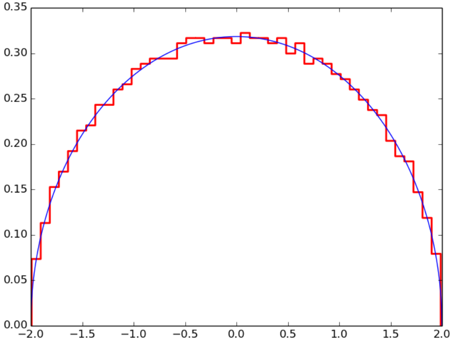
- Together with sample covariance matrices XX^T (e.g., where $\{X_{ij}\}_{ij}$ are iid) are a basic model of random matrix theory
- Used as "null" statistical models or pure noise (spiked models $W + vv^T$ are well-studied)
- Proposed by E. Wigner to model distribution of energy levels of complex quantum systems
- Universality of spectral quantities (eigenvalues, eigenvectors) in the limit $N \to \infty$ of interest to mathematicians

Universality: As $N \to \infty$ behavior of eigenvalues and eigenvectors does not depend on the choice of distribution of W_{ij}

Wigner's semicircle distribution,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(W)}(E)\mathrm{d}E = \frac{1}{2\pi}\sqrt{(4-E^2)_+}\mathrm{d}E =: \rho_{\mathrm{sc}}(E)\mathrm{d}E$$

almost surely.

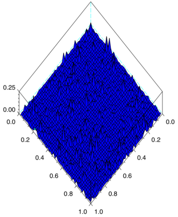


"Random matrix behavior" for local statistics:

Eigenvectors are delocalized:

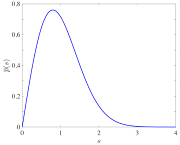
$$||u_i||_\infty \lesssim N^{-1/2}$$

(Haar-distributed on $O(N)$ for GOE)



Wigner surmise* for eigenvalue **gaps**:

$$\mathbb{P}[N(\lambda_{i+1} - \lambda_i) \in \mathrm{d}s] \sim \frac{\pi s}{2} \mathrm{e}^{-\pi s^2/4}$$



Tracy-Widom distribution for **extremal** eigenvalues:

$$\lim_{N\to\infty} N^{2/3}(\lambda_1 - 2) = \mathsf{TW}_1$$

- Limiting distributions first found for Gaussian ensembles using explicit formulas
- **Universality** for general Wigner matrices proven by many authors [Erdős-Schlein-Yau-Yin, Tao-Vu, Soshnikov, Péché...]

Models of **sparse** random matrices:

- Let $B = (B_{ij})_{i \leqslant j}$ be a symmetric matrix of iid Bernoulli random variables with $\mathbb{P}[B_{ij} = 1] = p$, with $p = (p_N)_N$.
- **Sparse Wigner matrices**: Hadamard product $H = \frac{1}{\sqrt{p}} B \circ W$, given by $H_{ij} = \frac{1}{\sqrt{p}} B_{ij} W_{ij}$.
- **Scaled and centered Erdős-Rényi adjacency matrices**: $A = \frac{1}{\sqrt{Np}} B$, and $\tilde{A} = A - \mathbb{E}[A]$. (Note that B is the adjacency matrix of $G(N, p)$.)

Notation: $d = pN$, is expected degree of a vertex in $G(N, p)$. d is the mean number of non-zero entries in a row of H.

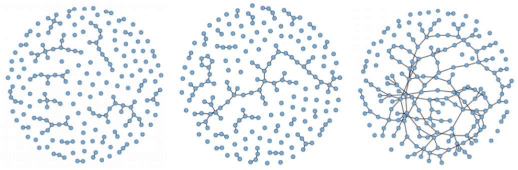
Scaling:

$$\mathbb{E}[\sum_{i=1}^{N} \lambda_i (B \circ W)^2] = \mathbb{E}[\mathrm{Tr}(B \circ W)^2] = \sum_{i,j} \mathbb{E}[B_{ij}^2 W_{ij}^2] = N^2 (pN^{-1}) = pN.$$

- We rescale by $\frac{1}{\sqrt{p}}$ to get a matrix with N eigenvalues that are each $\mathcal{O}(1)$

Questions about sparse random matrices:

- For what range of p do random matrix statistics survive (i.e., the limiting distributions found for Wigner matrices)?
- If not, what new spectral phenomena emerge?
- What is the behavior at the spectral bulk and spectral edges?
- What is the origin of any new behavior?



$p < p_c$ $\qquad$ $p = p_c$ $\qquad$ $p > p_c$

Percolation transition at $p_c = \frac{1}{N}$:

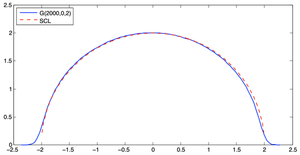
- For $d = Np < 1$ all connected components of $G(N, p)$ are small
- For $d = Np > 1$, a giant component emerges containing a macroscopic fraction of the vertices.

Second transition at $pN = \log(N)$, where *all* vertices are in the giant component.

Image: HSE

Limit of large degree $d = Np \to \infty$

As long as $d = Np \to \infty$ as $N \to \infty$, one recovers the semicircle distribution. Follows from original proof of Wigner, however *rate of convergence* is slower than for Wigner matrices.



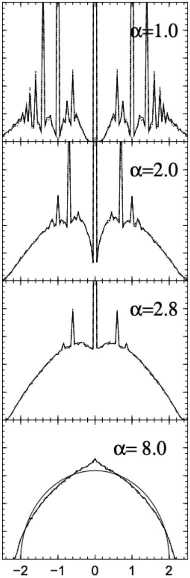
- If $d = Np \to \infty$ at any polynomial rate, then universality holds for eigenvalues in the bulk (i.e., Wigner surmise for $N(\lambda_{i+1} - \lambda_i)$) [Huang-L.-Yau, 2015]
- All eigenvectors are delocalized as long as $d = Np \geqslant C \log(N)$ [He-Knowles-Marcozzi, 2018].

Results hold for both sparse Wigner matrices and adjacency matrices of sparse random graphs such as $G(N, p)$.

Can allow for inhomogeneous graphs (block models, etc.) and d-regular graphs [Bauerschmidt-Knowles-Huang-Yau].

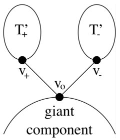
Image: Vu et. al

Limiting spectral measure for $G(N,p)$ with $d = Np = \alpha$ fixed



- No explicit formula for limit
- Density of states conjectured to extend past ± 2 as $p(E) \sim E^{-2\alpha E}$
- Continuous part iff $\alpha > 1$ [Bordenave-Sen-Virag, 2013]. $E = 0$ is in the continuous spectrum iff $\alpha > e$ [Coste-Salez, 2018]
- Mobility edge conjectured for $\alpha > 1.4$; emergence of delocalization and random matrix statistics

- Dense point spectrum
- Delta functions from disconnected finite trees as well as those grafted onto giant component. Mass is exponentially suppressed.



Images: Bauer, Golinelli

Behavior of extremal eigenvalues for slowly growing $d = pN$

Let $d = pN = b\log(N)$. There is a b_* such that [Knowles, et. al, 2019]:

- If $b > b_*$, then the extremal eigenvalues converge to the edge of the semicircle distribution ± 2
- If $b < b_*$, then the extremal eigenvalues are order $\sqrt{\log(N)}$.
- Exact transition is model-dependent. In the case of Erdős-Rényi, $b_* = \frac{1}{\log(4)-1}$ (different from the second transition).

Extremal eigenvalues arise from vertices x of largest degree d_x. An elementary tree calculation shows an eigenvalue $\lambda > 2$ and localized eigenvector arise iff $d_x > 2d$.

Existence of b_* follows from degree distribution

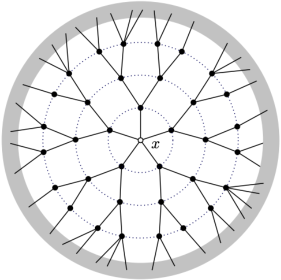


Image: Knowles et. al

Consider a vertex x of degree d_x, with each neighbor being the root of a d-regular tree. Let $\alpha_x = \frac{d_x}{d}$.

Apply Gram-Schmidt to A starting with 1_x. In the GS basis and in a neighbourhood of the vertex x, A can be written as M

$$M = \begin{pmatrix} 0 & \sqrt{\alpha_x} & & & & \\ \sqrt{\alpha_x} & 0 & 1 & & & \\ & 1 & 0 & 1 & & \dots \\ & & 1 & 0 & 1 & \\ & & \vdots & & & \end{pmatrix}$$

The vector $u_0 = 1$,

$$u_1 = \left(\frac{\alpha_x}{\alpha_x - 1}\right)^{1/2}, \qquad u_{i+1} = \left(\frac{1}{\alpha_x - 1}\right)^{1/2} u_i$$

is an eigenvector for eigenvalue $\frac{\alpha_x}{\sqrt{\alpha_x - 1}}$, and decreases exponentially iff $\frac{d_x}{d} = \alpha_x > 2$.

Tracy-Widom distribution for moderately growing d

In the case that the eigenvalues converge to the edges of the semicircle distribution do we have the $N^{-2/3}$ Tracy-Widom fluctuations?

Theorem. [Erdős-Knowles-Yau-Yin, 2011]. Let $d = pN \gg N^{2/3}$. Then,

$$\lim_{N\to\infty} N^{2/3}(\lambda_1 - 2) = \mathrm{TW}_1$$

Theorem. [Lee-Schnelli, 2017]. Let $d = pN \gg N^{1/3}$. Let,

$$L = 2 + \frac{s_4}{Np}$$

where s_4 is the fourth cumulant of the matrix entries. Then,

$$\lim_{N\to\infty} N^{2/3}(\lambda_1 - L) = \mathrm{TW}_1.$$

- The correction $\frac{s_4}{Np} \gg N^{-2/3}$, as soon as $d \ll N^{2/3}$
- Formula for L agrees with nonrigorous expansion of DoS of [Rodgers-Dominicis, 1988].

Theorem. [Huang-L.-Yau, 2019]. Let $d = pN \gg N^{2/9}$. Define,

$$L = 2 + \frac{s_4}{Np} + \frac{s_6}{(Np)^2} - \frac{9}{4}\frac{s_4^2}{(Np)^2}, \qquad \chi := \frac{1}{N}\sum_{i,j}\left(H_{ij}^2 - \frac{1}{N}\right)$$

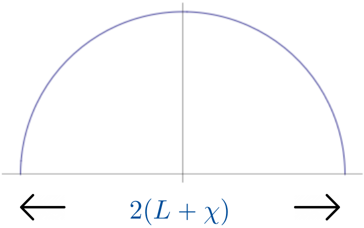
Then, we have the joint convergence of

$$\left(N^{2/3}(\lambda_1 - L - \chi), N\sqrt{p}\chi\right)$$

to independent TW_1 and $\mathcal{N}(0, s_4)$ random variables. In particular, for $d = cN^{1/3}$, $N^{2/3}(\lambda_1 - L)$ converges to $\mathrm{TW}_1 +$ Gaussian.

- For $d \ll N^{1/3}$ the fluctuations of $\chi \gg N^{-2/3}$
- The random variable χ can be interpreted as number of edges in ER graph
- For smaller d we expect more higher order deterministic and random quantities.

- χ can be interpreted as an extensive quantity measuring system size.
- Its fluctuations cause the semicircle density to be **stretched** (as opposed to shifted)
- In contrast to extremal eigenvalues of d-regular graphs, which are expected to have Tracy-Widom fluctuations down to $d = 3$.



- In particular, λ_1 shifts by $+\chi$ whereas λ_N shifts by $-\chi$.
- Can also be detected in the behavior of single eigenvalue fluctuations in the spectral bulk.

Consider eigenvalues in the bulk: we restrict to eigenvalue indices i so that $\varepsilon N \leqslant i \leqslant (1-\varepsilon N)$.

For Wigner matrices W,

$$\lim_{N\to\infty} \frac{N}{\sqrt{\log(N)}}(\lambda_i(W) - \mathbb{E}[\lambda_i(W)]) = \mathcal{N}(0,\sigma^2)$$

- First proven for Gaussian ensembles by [Gustavsson, O'Rourke].
- Extended to general Wigner matrices by [L.-Sosoe, Bourgade-Mody]

For sparse ensembles, *away from* $E=0$, [He, 2018]

$$\lim_{N\to\infty}(N\sqrt{p})(\lambda_i - \mathbb{E}[\lambda_i]) = \mathcal{N}(0,\hat{\sigma}^2).$$

Furthermore correlation of two eigenvalues is ± 1.

However, eigenvalue *gaps* retain the universal GOE fluctuations,

$$N(\lambda_{i+1}-\lambda_i) \sim \text{Wigner surmise.}$$

Dyson Brownian motion as the origin of microscopic fluctuations

How do we prove results about the microscopic fluctuations?

- Use *perturbation theory* to compare H with

$$H_t = H + \sqrt{t}G$$

  where G is a Gaussian matrix. This adds some "noise."
- H_t is the solution to a stochastic process known as Dyson Brownian motion: "Brownian motion in the space of matrices"
- Prove fast *mixing time* of H_t to the invariant Gaussian measure

In our example, H is a sparse matrix. It's fluctuations give rise to the Gaussian term, and G gives rise to the Tracy-Widom distribution.

In conclusion:

- As $p \to 0$ at faster rates as $N \to \infty$ rich spectral features emerge
- Spectral universality in the bulk is quite robust; holds as long as $pN \geqslant N^{\varepsilon}$.
- Extremal behavior is more complicated; transition at $pN = N^{1/3}$ from Tracy-Widom to Gaussian fluctuations

Thank you for your attention!