Products of thin random matrices and random walks in random media

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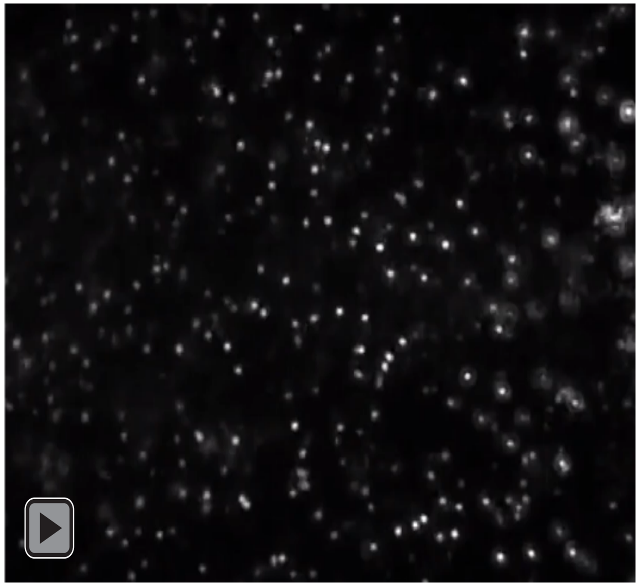
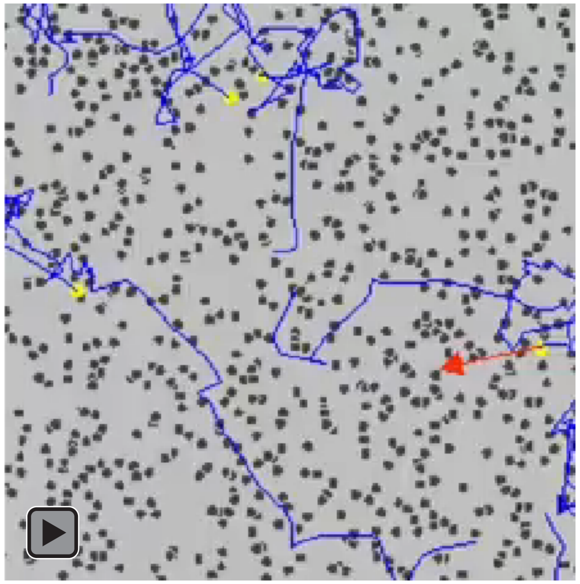
Products of random matrices

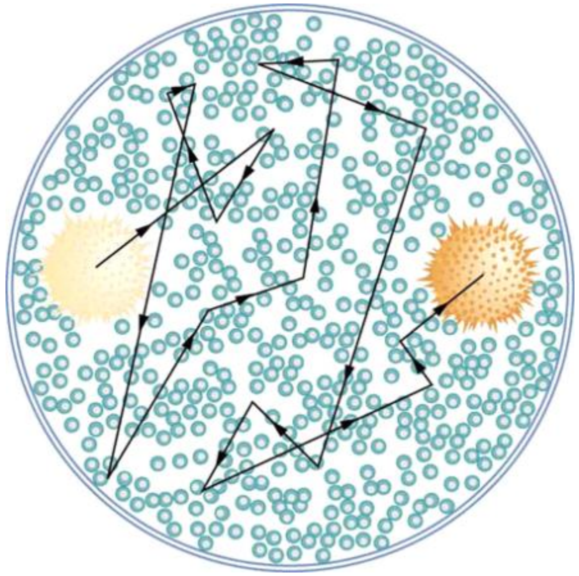
[Furstenberg, Kesten '60], [Cohen, Newman '84]... studied the Lyapunov exponents for products of independent iid entry matrices. This type of result informs the understanding of stability for systems of SDEs [May '72] and deep neural nets [e.g. recent work of Pennington, Hanin...].

Today, I will talk about products of **thin matrices**. Matrix entries encode transition probabilities for **random walks in random media** and they display a remarkable asymptotic fluctuation behavior.

'Brownian' motion



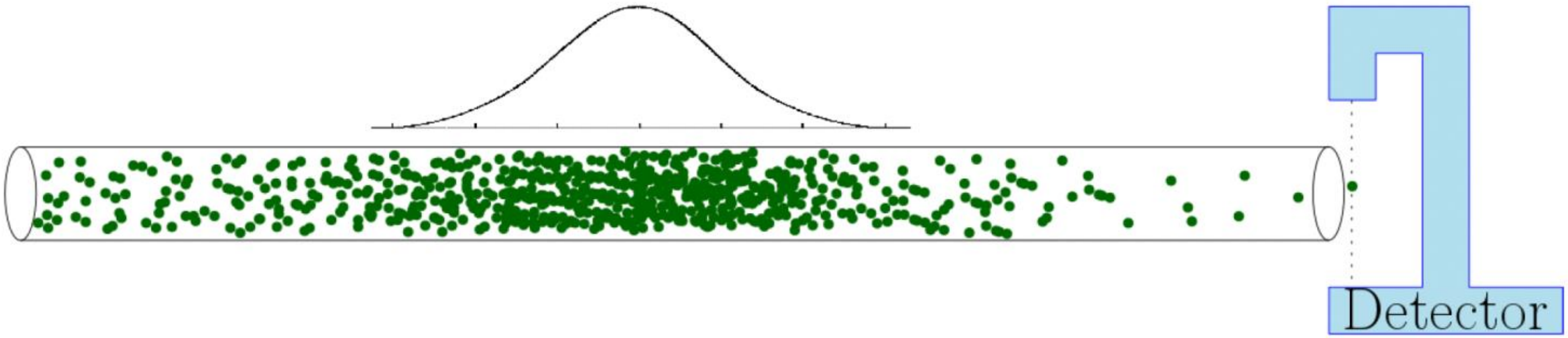




How do you <u>effectively</u> describe this motion?

- [Leeuwenhoek, 1600s] + [Brown, 1828]
- [Einstein, 1905] + [Smoluchowski, 1906]
- [Perrin, 1908]
- [Wiener, 1918]

How effective is this theory?



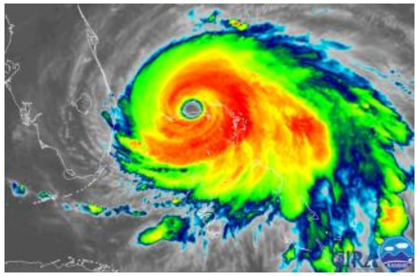
Compare <u>bulk</u> and <u>boundary</u> for:

1. Independent random walks,
2. Random walks in a same random environment.

Punchline: Bulk is as Einstein said in both cases.

Boundary behaves VERY DIFFERENTLY between them.

Independent random walks



LDP CLT LLN CLT LDP

Classical probability

$$\mathsf{P}(X_{t+1} = X_t + 1) = \frac{\alpha}{\alpha+\beta}, \qquad \mathsf{P}(X_{t+1} = X_t - 1) = \frac{\beta}{\alpha+\beta}$$

LLN: $\dfrac{X_t}{t} \longrightarrow \dfrac{\alpha-\beta}{\alpha+\beta}$

CLT: $\dfrac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0,1)$ with $\sigma = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta}$

LDP (Large deviations principle): For $\dfrac{\alpha-\beta}{\alpha+\beta} < x < 1$,

$$\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big)}{t} \longrightarrow -I(x)$$

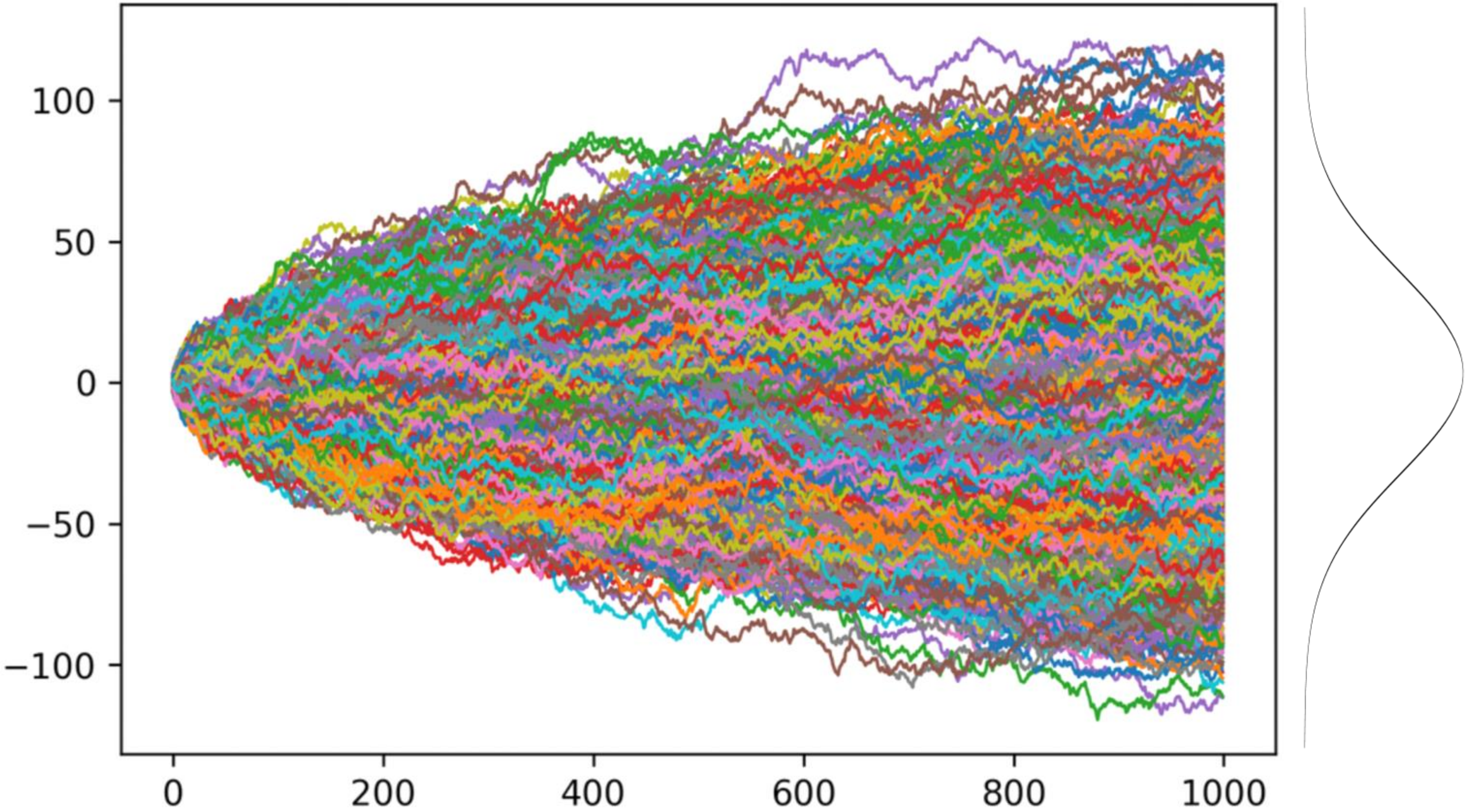
with $I(x) = \sup_{z\in\mathbb{R}}\big(zx - \lambda(z)\big)$ and $\lambda(z) := \log\Big(\mathsf{E}\big[e^{zX_1}\big]\Big)$

(e.g. for $\alpha = \beta,\ I(x) = \frac{1}{2}\Big((1+x)\log(1+x) + (1-x)\log(1-x)\Big)$)

The transition probability from x at time 0 to y at time t is the (x,y) entry of the t^{th} power of a tridiagonal matrix With $\dfrac{\alpha}{\alpha+\beta}$ and $\dfrac{\beta}{\alpha+\beta}$ off-diagonal and 0 on-diagonal.

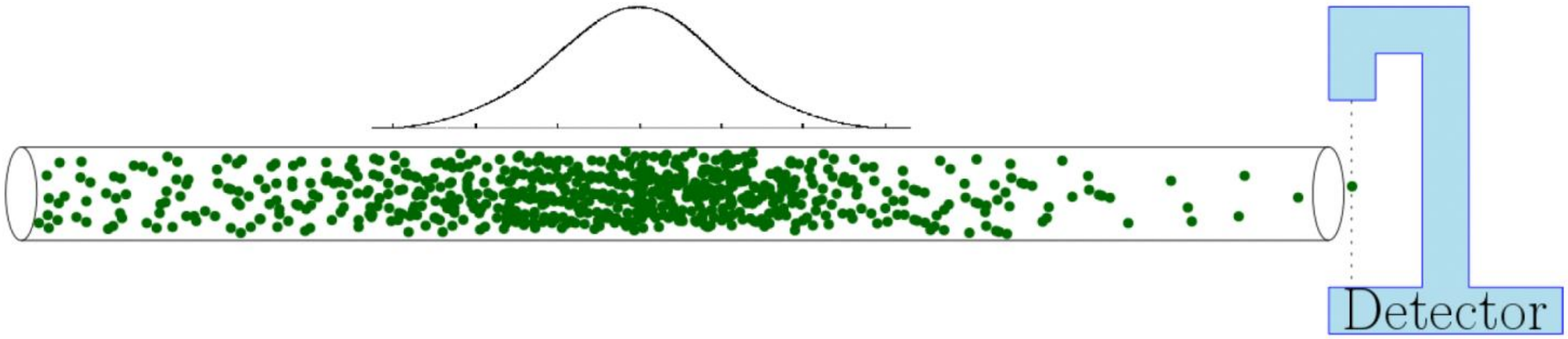
$$\begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 & 0 \\ \ddots & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}^t$$

Many independent random walks



$$Prob(\max(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(N)}) \leq x) = (1 - Prob(X_t > x))^N$$

What does this tell us?



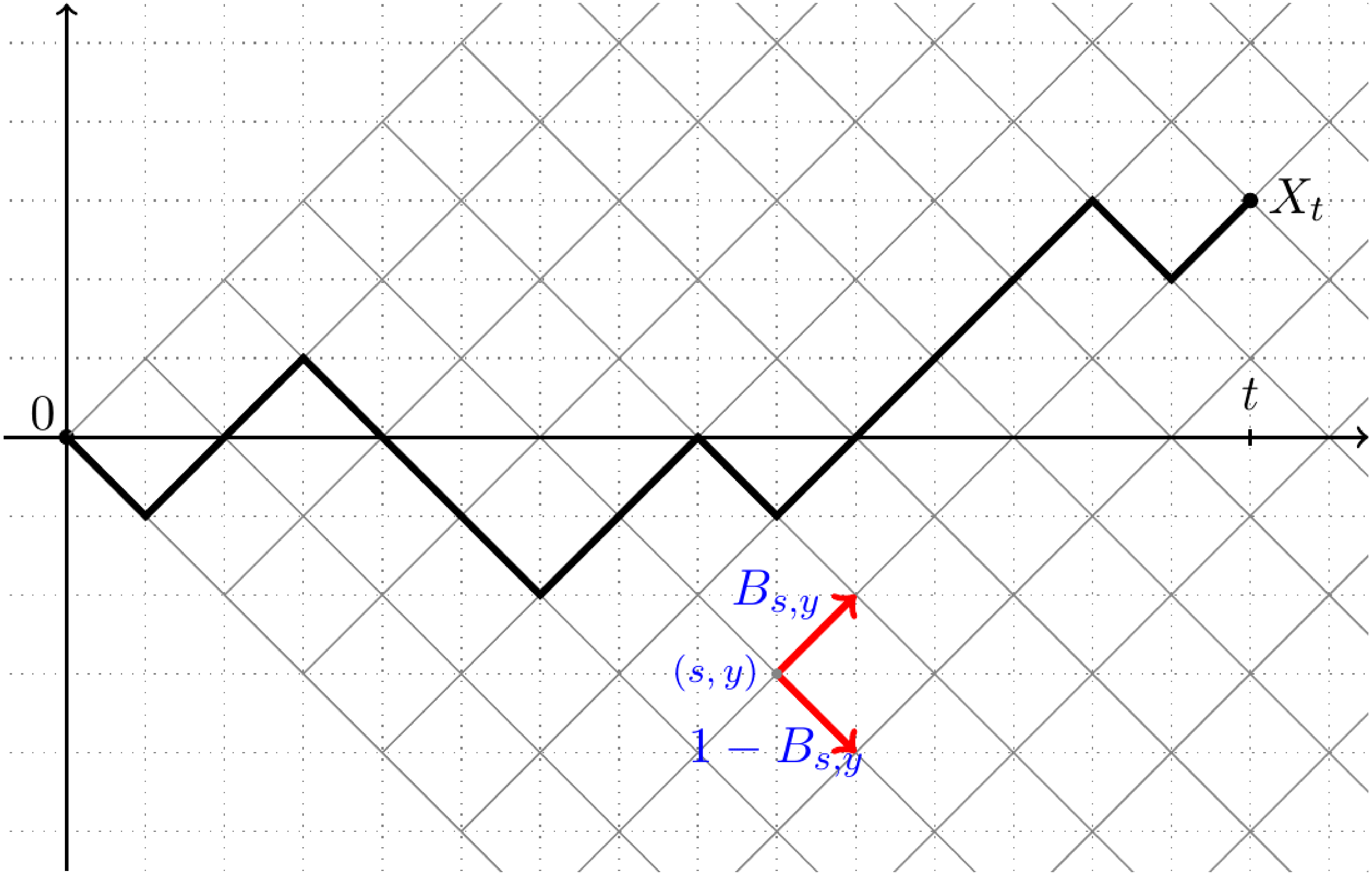
Independent random walks

- Bulk: Diffusion + Gaussian + heat equation.
- Boundary: For $N=e^{ct}$ independent random walks,

$\max(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(N)}) \sim c_1 t + c_2 \log(t) + c_3$ Gumbel

with $c_1=I^{-1}(c)$ and Gumbel distribution:

Random walk in random environment



Now off-diagonal entries for s^{th} matrix are $B_{s,y}$ and $1-B_{s,y}$.

Bulk versus boundary

Random walks have same LLN and CLT as when $B_{t,x}$ is replaced by $E[B_{t,x}]$. [Rassoul-Agha, Seppalainen, Yilmaz '04+'13]

LDP is strongly influenced by randomness of the $B_{t,x}$.

When $B_{t,x} \sim \text{Beta}(\alpha, \beta)$ we can exactly compute the distribution of the transition probabilities (or the matrix product entries) and study precise asymptotics.

For simplicity, we consider $\alpha = \beta = 1$ so $B_{t,x} \sim U[0,1]$.

Theorem [Corwin, Barraquand '15]:

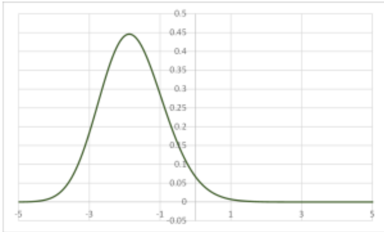
When $B_{t,x} \sim U[0,1]$ we have the following LDP:

$$\lim_{t\to\infty} -\frac{\log\left(\mathsf{P}_B(X_t > xt)\right)}{t} = I(x) = 1 - \sqrt{1-x^2}$$

Furthermore, the residual randomness manifests as

$$\frac{\log\left(\mathsf{P}_B(X_t > xt)\right) + I(x)t}{\sigma(x)\cdot t^{1/3}} \implies \mathcal{L}_{GUE}$$

$\sigma(x)^3 = \frac{2I(x)^2}{1-I(x)}$

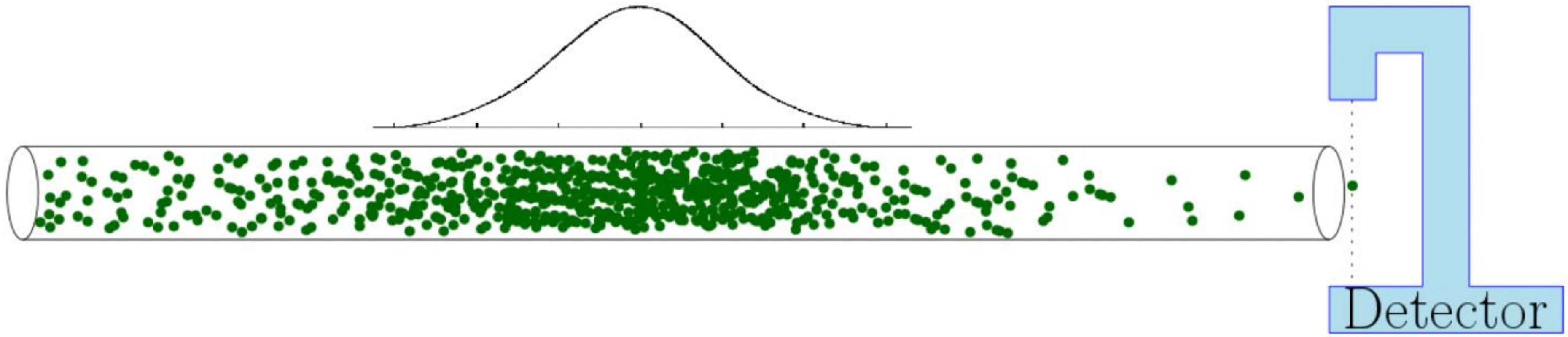


Equivalently: For $N=e^{ct}$ independent random walks in the same random environment, and for $c'_1=(1-(1-c)^2)^{1/2}$

$$\max(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(N)}) \sim c'_1\, t + c'_2\, t^{1/3}\, GUE$$

GUE Tracy-Widom distribution from another RMT problem!

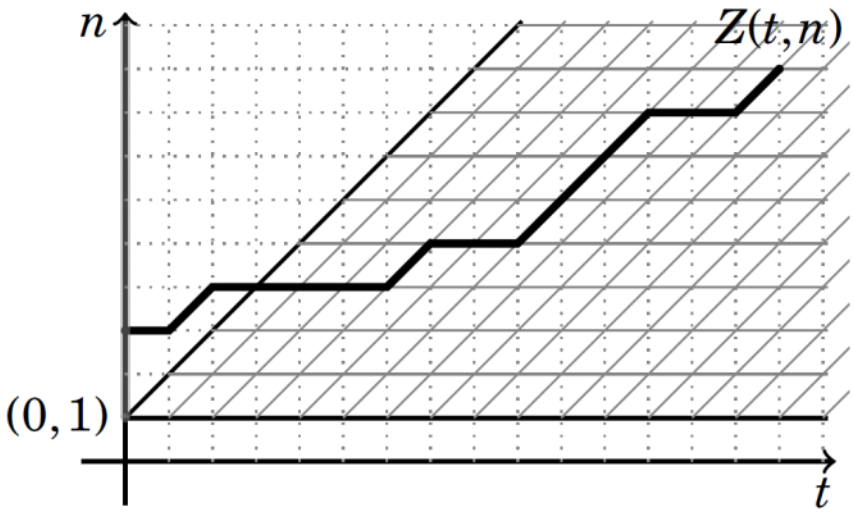
Back to the experiment

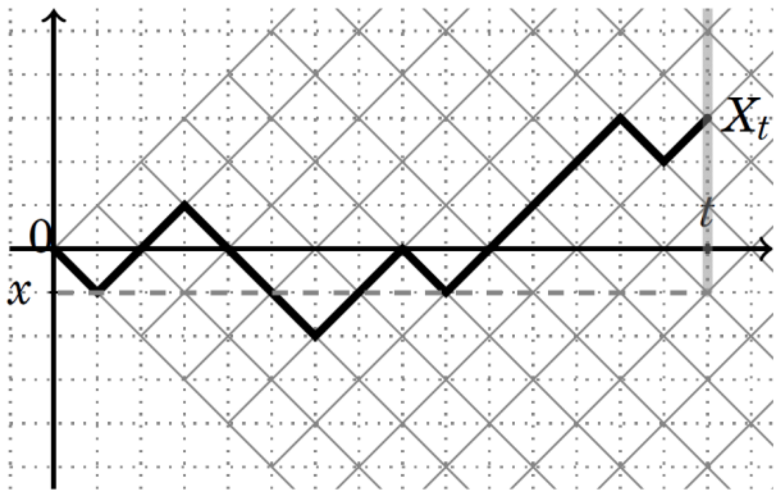


How might me 'test' whether a real system is better modeled by independent random walks, or random walks in a random environment?

-> The speed of the boundary is not universal, but the time$^{1/3}$ scaling and GUE statistic for the variance is believed to be quite universal!

Why can we do any of this





Prob($X_t > t-2n$) = $Z(t,n)$ where Z solves the recursion

$$Z(t,n) = B_{t,n} Z(t-1,n) + (1-B_{t,n}) Z(t-1,n-1), \quad Z(0,n) = 1_{n>0}$$

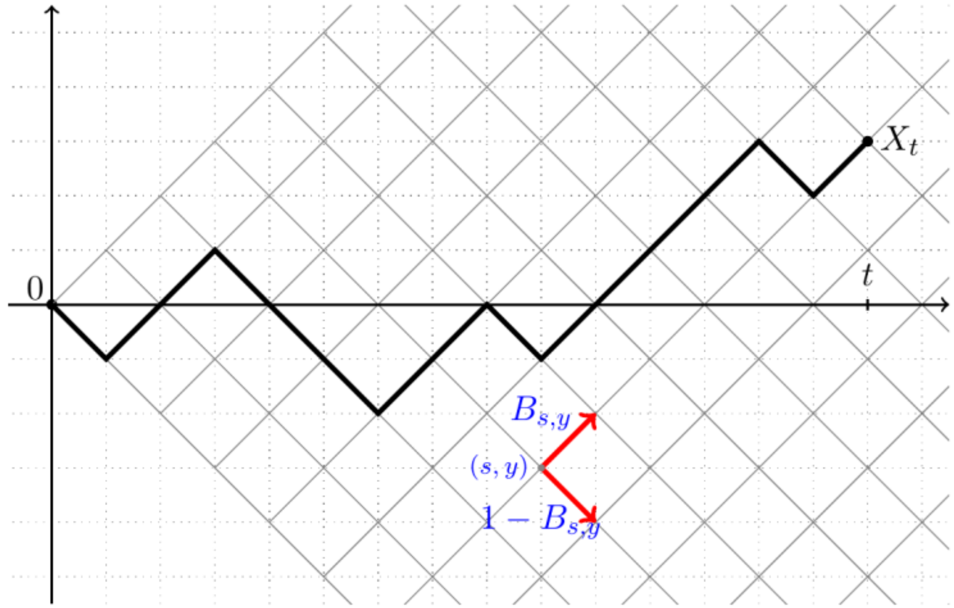
- For $B_{t,n}$ deterministic, Fourier analysis solves this.
- For $B_{t,n}$ random, no such luck! But, mixed-moments satisfy Bethe ansatz solvable many-body systems…

Back to the matrices

Considered $M_1 \cdots M_t$ when the B's are iid Beta-distributed.

$$M_t := \begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 & 0 \\ \ddots & 0 & B_{t,x-1} & 0 & 0 & 0 \\ 0 & 1-B_{t,x} & 0 & B_{t,x} & 0 & 0 \\ 0 & 0 & 1-B_{t,x+1} & 0 & B_{t,x+1} & 0 \\ 0 & 0 & 0 & 1-B_{t,x+2} & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

Showed that entries of the product distance $O(t)$ from the diagonal decay with an exponential rate with fluctuations of order $t^{1/3}$ and GUE distribution.

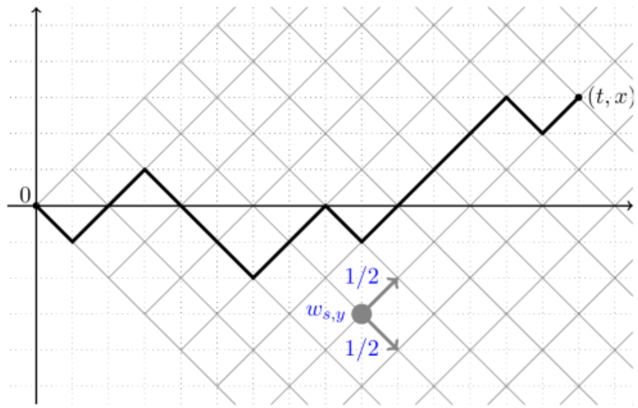


$M_1 \quad \cdots \quad M_t$

Another solvable model

Now consider $\bar{M}D_1 \cdots \bar{M}D_t$ where
$$\bar{M} := \begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 & 0 \\ \ddots & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$
and $D_t = \mathrm{diag}(\cdots, w_{t,x-1}, w_{t,x}, w_{t,x+1}, \cdots)$.

The matrix entries are sums over paths of products of vertex weights along the paths (directed polymer partition functions).



If the weights are independent and inverse-Gamma distributed, this model is also exactly solvable with similar asymptotic results as in the previous example [C, O'Connell, Seppalainen, Zygouras '12], [Borodin, C, Remenik '13]

And beyond?

The 'Kardar-Parisi-Zhang' behavior seen in these two exactly solvable examples should be very universal:

- Products of thin random matrices with short-range dependence, should have matrix entries which enjoy the same types of asymptotic behaviors.

I focused on a connection to random walks in random environments. There are many other systems related to such products (interface growth models, particle systems, vertex models,...).

Maybe you can come up with some new examples???